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Phenomenological Theory of Multi-Mode Surface Wave Excitation, Propagation and Diffraction

I. Plane Structures

SAMUEL N. KARP and FRANK C. KARAL, JR.

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1. Introduction

It is well known that surface waves can be excited and propagated along particular idealized electromagnetic structures. Examples are dielectric coated surfaces, dielectric slabs and corrugated or laminated structures.* In so far as the surface wave feature is concerned, it is known that the details of the structure may be suppressed and replaced by an impedance boundary condition [9]. Subsequent developments showed that the use of impedance boundary conditions facilitated the solution of more complex problems such as the excitation, propagation and diffraction of surface waves on structures with discontinuities in impedance and geometry [5,6, 15-24]. The phenomenological representation by an impedance boundary condition is also useful in analyzing the effect of other structures such as absorbers and surfaces with anisotropic conductivity [6, 16].

Since an open surface structure acts to some extent like a waveguide, and since its theory is of use in surface wave antennas, it would seem to be of interest to generalize the existing theory to multi-mode surface wave guides. This means to generalize the body of theory mentioned above to the case of surfaces that support more than one surface wave. Thus one should ultimately possess a body of theory for surface wave guides analogous to that already available for conventional wave guides. Typical problems of interest are the analysis of the effects of

* The Bibliography given at the end of the paper is representative but not complete. Additional references are given by Collin [7], Cottony [8] and Zucker [25].

radiation from junctions, changes in cross section, effects of terminations, etc. The present paper is the first part of a series devoted to these topics.

The phenomenological theory on which the generalization is based has not been verified experimentally. However, it is clear that such surfaces can be realized, at least by sufficiently thick dielectric slabs or coatings. There are also certain qualitative features of the results for the phenomenological theory of one mode in the case of the excitation problem which are in agreement with the exact theoretical results for a slab or thin coating. It would certainly be worthwhile to compare the exact theory with the phenomenological theory presented here with more than one mode and in detail. Also, experimental results for one or more modes would also be desirable. The phenomenological theory offers the possibility of analytical solutions of complex problems whose analysis from the standpoint of a given detailed structure is difficult or impossible by present analytical methods. Another problem is the correspondence between detailed properties of the structure and the choice of a generalized boundary condition. The problem has been treated for a particular structure which support one surface wave. [1]. It is also well understood in the case of a dielectric structure for one mode and has been partially analysed by the authors (in a paper to appear shortly) for a multi-mode dielectric structure. In brief, it would seem desirable to compare the phenomenological theory (i.e, exact solutions using generalized boundary conditions) with exact solutions in which the detailed structure is taken into account more explicitly.

The purpose of the present paper is to study the electromagnetic field that arises when a magnetic line dipole source is located above a plane structure that supports several surface waves if the surface of the structure is characterized by a generalized impedance boundary condition. This condition is introduced here for the first time. We give an exact solution for plane structures supporting one, two and three surface waves. We obtain the magnitudes of the surface waves and give simple formulas for the radiated far field patterns. We also show how the methods employed in this paper can be extended to the case of plane structures that support any number of surface waves. It should be emphasized that the effects of terminations, finiteness, etc., which constitute the main applications of the present formulation, are excluded from consideration in this introductory paper. The treatment of these basic diffraction effects, along the lines of [5, 6, 15-24], are the subject of subsequent papers.

The boundary condition we employ to describe the propagation of several surface waves along a plane surface wave structure is given by

$$\prod_{i=1}^n \left(\frac{\partial}{\partial y} + \lambda_i \right) u(x, y) = 0 \quad , \quad y = 0 \quad (1.1)$$

where $u(x, y)$ is the z-component of the magnetic vector $\vec{H}(x, y)$, x and y are the usual cartesian rectangular coordinates, and λ_i are constants characteristic of the surface. The value of λ_i is given by

$$\lambda_i = i\omega\epsilon Z_i = i\omega\epsilon(R_i - iX_i) \quad , \quad (1.2)$$

where ϵ is the permittivity of free space, ω is the angular frequency, and Z_i , R_i and X_i are the impedances, resistances and reactances of the surface, respectively. For surface wave propagation it is necessary to require that*

$$\text{Re } \lambda_i > 0 \quad . \quad (1.3)$$

* This assumption is employed in the illustrative examples that follow, but is not an essential restriction.

It is worth noting that if there is to be negligible attenuation of the surface waves along the surface, it is necessary to require that $R_1 \ll X_1$. This boundary condition is a natural generalization of the usual impedance boundary condition employed on surfaces supporting one surface wave. For example, when $n = 1$ in (1.1) we obtain the boundary condition

$$\left(\frac{\partial}{\partial y} + \lambda_1\right)u(x,y) = 0, \quad y = 0. \quad (1.3)$$

This boundary condition is well known and is a good approximation to several important physical configurations. Perhaps the simplest example is a thin dielectric layer lying on a perfectly conducting ground plane. This and other possible physical configurations are discussed in [7], where additional references are given. It is clear that a surface wave solution of the form

$$u_s(x,y) = A_1 e^{-\lambda_1 y + i\sqrt{k^2 + \lambda_1^2} |x|} \quad (1.4)$$

satisfies the wave equation, is outgoing at infinity, and satisfies (1.3).

Hence a surface characterized by (1.4) can indeed support surface waves.

Similarly, the generalized boundary condition (1.1) can also support surface waves of the form

$$u_s(x,y) = \sum_{i=1}^n A_i e^{-\lambda_i y + i\sqrt{k^2 + \lambda_i^2} |x|}. \quad (1.5)$$

Therefore (1.1) is the natural extension to structures supporting several surface waves. We shall call these multi-mode surface wave structures and

use the generalized boundary condition (1.1) as their defining property. We will not discuss the physical realizability of the surface wave structures here. We point out, however, that a perfectly conducting ground plane with a sufficiently thick dielectric coating, or several different thin dielectric coatings, will support more than one surface wave.

In Section 2 we study the electromagnetic field that arises when a magnetic line dipole source is located above a plane structure that supports a single surface wave. In the special case when the line source is on the surface, the method and result reduce to those given previously in [20]. Nevertheless, the treatment is complete and given in some detail since the procedure is the same when more surface wave modes are considered. The basic ideas for obtaining the exact solution and the calculation of the surface waves are contained in this section.

In Sections 3 and 4 we study the electromagnetic fields that arise when two and three surface waves are excited on a plane structure, respectively. The fields in both cases are produced by a line magnetic dipole located above the surface. In Section 5 we consider the case of any number of modes and summarize the main steps for treating this problem. In addition to the surface waves, the radiated far field is of considerable interest. In Section 6 we determine the far field radiation patterns for one, two and three modes and present them in a form suitable for computation. The method is direct and employs an elementary procedure that does not require the exact results obtained in the earlier sections. We also show how to find the radiation patterns for any number of modes.

2. Single-Mode Surface Wave Structure

We first study the electromagnetic field that arises when a magnetic line dipole source is located above a plane structure that supports a single surface wave. Consider the plane surface $y = 0$ and suppose a magnetic line dipole is located at the point $x = 0, y = y_0$. See Figure 1. In the region $y \geq 0$ we assume that we have free space. The boundary condition on the surface is given by the impedance condition

$$\left(\frac{\partial}{\partial y} + \lambda_1\right)u(x,y) = 0, \quad y = 0 \quad (2.1)$$

where $u(x,y)$ is the magnetic field component H_z and λ_1 is a constant characteristic of the surface material. We wish to solve the time-reduced Maxwell's equations, subject to the prescribed condition, and obtain the amplitude of the surface waves propagated along the surface.

The time dependent form of Maxwell's equations is

$$\left. \begin{aligned} \nabla \times \vec{H} &= -i\omega\epsilon \vec{E} \\ \nabla \times \vec{E} &= i\omega\mu \vec{H} \end{aligned} \right\} \quad (2.2)$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic field intensities, and ϵ and μ are the permittivity and magnetic permeability of free space. We assume the time dependence to be of the form $e^{-i\omega t}$. Because of the geometry, the field produced is independent of z and hence is completely determined by the value of $H_z \equiv u(x,y)$. We have

$$H_x = H_y = E_z = 0 \quad (2.3)$$

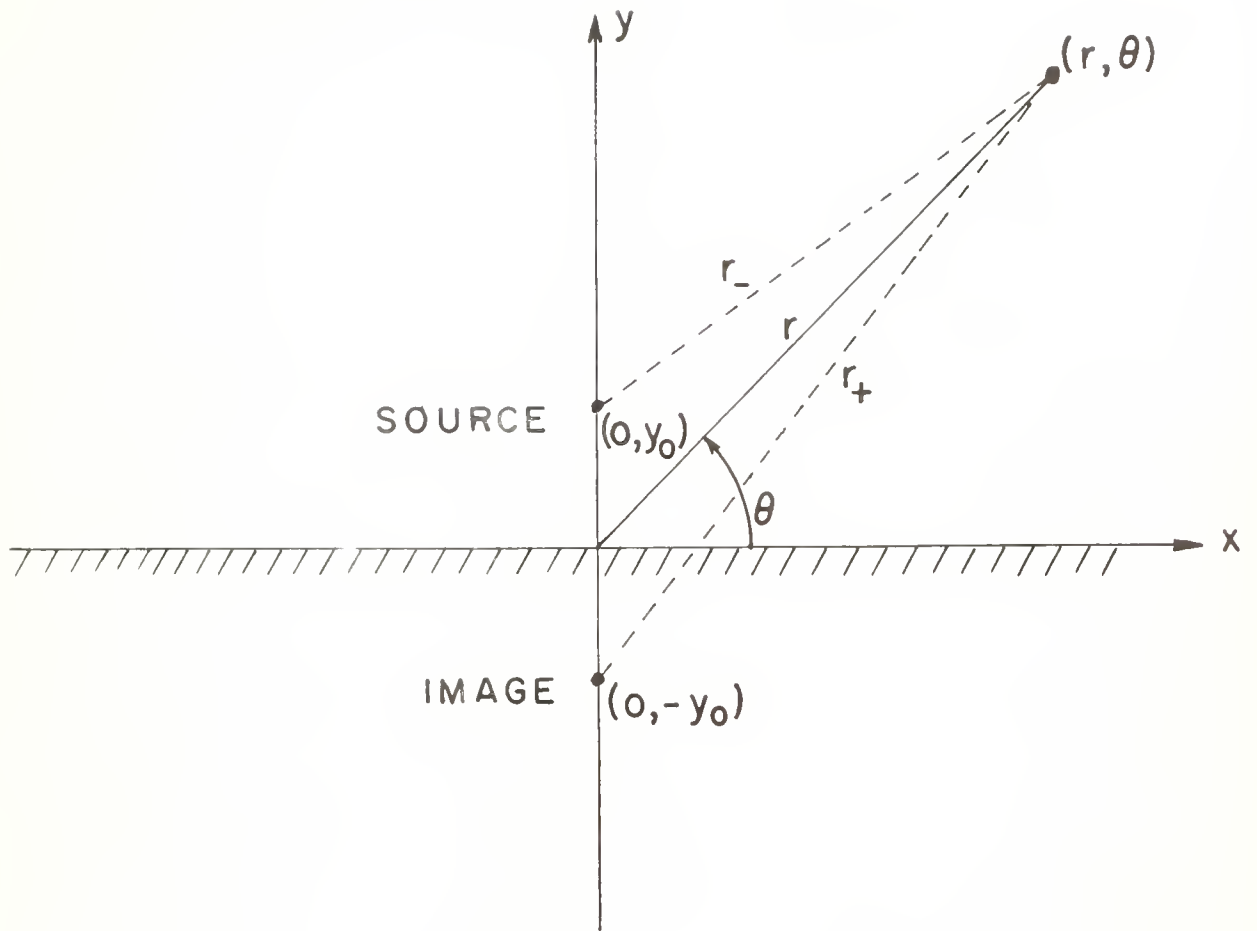


Figure 1. Magnetic Line Dipole Source and Image in the Presence of a Surface Characterized by a Generalized Impedance Boundary Condition.

and

$$\left. \begin{aligned} E_x &= -\frac{1}{i\omega\epsilon} \frac{\partial u}{\partial y} \\ E_y &= +\frac{1}{i\omega\epsilon} \frac{\partial u}{\partial x} \end{aligned} \right\} \quad (2.4)$$

The function $u \equiv u(x, y)$ satisfies the wave equation

$$(\nabla^2 + k^2)u = -4\pi b(x-0)\delta(y-y_0) \quad (2.5)$$

where ∇^2 is the rectangular Laplacian, k is the propagation constant of free space and δ is the Dirac delta function. Therefore, the mathematical problem reduces to that of solving the inhomogeneous wave equation (2.5) subject to the mixed boundary condition given by (2.1). In addition to the prescribed boundary condition, we require the far field to be outgoing, and the value of the total magnetic field(excluding the source) to be finite everywhere.

We now employ a technique developed by the authors [6,8,10-12] for use in solving mixed boundary value problems arising in electromagnetic diffraction theory. Let us first introduce an auxiliary function v defined by

$$v = \left(\frac{\partial}{\partial y} + \lambda_1\right)u \quad (2.6)$$

Then v satisfies the equation

$$(\nabla^2 + k^2)v = -4\pi\left(-\frac{\partial}{\partial y_0} + \lambda_1\right)\delta(x-0)\delta(y-y_0) \quad (2.7)$$

where we have used the result $\frac{\partial}{\partial y}\delta(y-y_0) = -\frac{\partial}{\partial y_0}\delta(y-y_0)$. Also, v is now subject to the simpler conditions

$$\left. \begin{array}{l} (1) \quad v = 0 \quad , \quad y = 0 \\ (2) \quad \text{Outgoing traveling waves at } \infty \end{array} \right\} . \quad (2.8)$$

If the problem for v can be solved, it is possible to find a solution for the problem involving u , since the differential equation (2.6) can be integrated. A particular solution of (2.6) is

$$u_p(x,y) = e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} v(x,\eta) d\eta \quad (2.9)$$

The solution of the auxiliary problem is easily constructed by introducing the image source located at the point $x = 0$, $y = y_0$. The desired solution for v is

$$v(x,y) = i\pi \left(-\frac{\partial}{\partial y_0} + \lambda_1 \right) \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\} \quad (2.10)$$

where

$$r_{\pm} = \sqrt{x^2 + (y_{\pm} - y_0)^2} . \quad (2.11)$$

Here r_- and r_+ are the distances from the source and image to the observation point, respectively. Clearly the solution vanishes on the boundary, is outgoing at infinity, and possesses the correct singular behavior at the source $x = 0$, $y = y_0$. If we substitute (2.10) into (2.9) and integrate by parts we obtain

$$u_p(x,y) = i\pi \left\{ H_0^{(1)}(kr_-) + H_0^{(1)}(kr_+) \right\} - 2\pi i \lambda_1 e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_0^{(1)}(kR_+) d\eta \quad (2.12)$$

where R_+ is the same as r_+ except that y is replaced by the integration variable

η . The function $u_p(x,y)$ just found satisfies the original boundary condition (2.1), possesses the correct source behavior and is outgoing at ∞ . However, it is not a wave function since the x -derivative is discontinuous in regions including the positive y -axis. This defect can be remedied, as will be shown below, by adding to (2.12) another solution of the wave equation as follows:

$$u(x,y) = i\pi \left\{ H_0^{(1)}(kr_-) + H_0^{(1)}(kr_+) \right\} - \pi i \lambda_1 e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_0^{(1)}(kR_+) d\eta + A_1 e^{-\lambda_1 y + i\sqrt{k^2 - \lambda_1^2} |x|} \quad (2.13)$$

The introduction of the additional term with the unknown constant A_1 now makes it possible to satisfy the continuity condition involving $\frac{\partial u}{\partial x}$. We note first that the form of the additional term represents a surface wave. Hence, when we impose the continuity requirement on $\frac{\partial u}{\partial x}$ and evaluate A_1 , we obtain the amplitude of the surface wave propagated away from the source. This is the dominant part of the field at large distances, as will be seen later.

We now proceed to determine the value of A_1 . The continuity requirement of $\frac{\partial u}{\partial x}$ across the line $x = 0$ can be expressed as the following jump condition:

$$\left[\frac{\partial u}{\partial x} \right] = \lim_{x \rightarrow 0} \left\{ \frac{\partial u}{\partial x} (x > 0, y) - \frac{\partial u}{\partial x} (x < 0, y) \right\} = 0 \quad (2.14)$$

From (2.13) we obtain

$$\begin{aligned} \frac{\partial u}{\partial x} = & i\pi \frac{\partial}{\partial x} \left\{ H_0^{(1)}(kr_-) + H_0^{(1)}(kr_+) \right\} + 2\pi i k \lambda_1 e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_0^{(1)}(kR_+) \frac{x}{R_+} d\eta \\ & + iA_1 \sqrt{k^2 + \lambda_1^2} \operatorname{sgn} x e^{-\lambda_1 y + i\sqrt{k^2 + \lambda_1^2} |x|} \end{aligned} \quad (2.15)$$

where we have used the identity

$$\frac{\partial}{\partial x} H_0^{(1)}(kr) = -k H_1^{(1)}(kr) \cos \theta \quad . \quad (2.16)$$

Substituting (2.15) into (2.14), and noting that the first terms for $\frac{\partial u}{\partial x}$ yield a zero contribution to the jump, we have

$$\begin{aligned} \left[\frac{\partial u}{\partial x} \right] = & \lim_{x \rightarrow 0} \left\{ 4\pi i k \lambda_1 e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_1^{(1)}(kR_+) \frac{x}{R_+} d\eta \right. \\ & \left. + 2iA_1 \sqrt{k^2 + \lambda_1^2} e^{-\lambda_1 y + i\sqrt{k^2 + \lambda_1^2} |x|} \right\} . \end{aligned} \quad (2.17)$$

The limit of the integral on the right can be evaluated by observing that the main contribution occurs in the neighborhood of $\eta = -y_0$. A straight forward calculation yields

$$\lim_{x \rightarrow 0} \int_{-\infty}^y e^{\lambda_1 \eta} H_1^{(1)}(kR_+) \frac{x}{R_+} d\eta = -\frac{2i}{k} e^{-\lambda_1 y_0} \quad (2.18)$$

Using this result in (2.17) we find

$$A_1 = \frac{4\pi i \lambda_1}{\sqrt{k^2 + \lambda_1^2}} e^{-\lambda_1 y_0} \quad . \quad (2.19)$$

This is the desired amplitude for the surface wave. The exact solution of the problem is given by

$$u(x,y) = i\pi \left\{ H_0^{(1)}(kr_-) + H_0^{(1)}(kr_+) \right\} - \pi i \lambda_1 e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_0^{(1)}(kR_+) d\eta \\ + \frac{4\pi i \lambda_1}{\sqrt{k^2 + \lambda_1^2}} e^{-\lambda_1 y_0} e^{-\lambda_1 y + i\sqrt{k^2 + \lambda_1^2} |x|} \quad (2.20)$$

Note the form of the integral

$$I = \lambda_1 e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_0^{(1)}(k\sqrt{x^2 + (\eta + y_0)^2}) d\eta. \quad (2.21)$$

This integral is basic in the present paper. In fact, the solution for the multi-mode problem is a linear combination of the terms in (2.20) with different decay constants λ_i in the integral and surface wave.

3. Two-Mode Surface Wave Structure

We next study the electromagnetic field that arises when a magnetic line dipole source is located above a plane structure that supports two surface waves. Consider the plane surface $y = 0$ and suppose a magnetic line dipole is located at the point $x = 0$, $y = y_0$. See Figure 1. The boundary condition on the surface is given by the impedance condition

$$\left(\frac{\partial}{\partial y} + \lambda_1\right)\left(\frac{\partial}{\partial y} + \lambda_2\right)u(x,y) = 0 \quad , \quad y = 0 \quad (3.1)$$

where u is the magnetic field component H_z , λ_1 and λ_2 are constants characteristic of the surface material, and $\lambda_1 \neq \lambda_2$. We wish to solve the time-reduced Maxwell's equations, subject to the prescribed condition, and obtain the amplitudes of the surface waves propagated along the surface. The magnetic field component $H_z = u(x,y)$ satisfies the wave equation given by (2.5). In addition to the prescribed boundary condition (3.1), we require the far field to be outgoing and the value of the total magnetic-field (excluding the source) to be finite everywhere. Note that once u is known, the electric field components can be determined from (2.4).

We now follow the same procedure as was used in Section 2.

Let

$$v = \left(\frac{\partial}{\partial y} + \lambda_1\right)\left(\frac{\partial}{\partial y} + \lambda_2\right) u. \quad (3.2)$$

Then v satisfies the equation

$$(\nabla^2 + k^2)v = -4\pi \left[\frac{\partial^2}{\partial y_0^2} - (\lambda_1 + \lambda_2) \frac{\partial}{\partial y_0} + \lambda_1 \lambda_2 \right] \delta(x-0)\delta(y-y_0) \quad (3.3)$$

subject to the simpler conditions

$$\left. \begin{array}{ll} (1) & v = 0 \quad , \quad y = 0 \\ (2) & \text{Outgoing traveling waves at } \infty \end{array} \right\} . \quad (3.4)$$

A particular solution of (3.2) is

$$u_p(x,y) = \frac{e^{-\lambda_1 y}}{\lambda_2 - \lambda_1} \int_{-\infty}^y e^{\lambda_1 \eta} v(x,\eta) d\eta + \frac{e^{-\lambda_2 y}}{\lambda_1 - \lambda_2} \int_{-\infty}^y e^{\lambda_2 \eta} v(x,\eta) d\eta. \quad (3.3)$$

The solution of the auxiliary problem is easily constructed, as before, by introducing the image of the source. The desired solution for v is

$$v(x,y) = i\pi \left[\frac{\partial}{\partial y_0} - (\lambda_1 + \lambda_2) \frac{\partial}{\partial y_0} + \lambda_1 \lambda_2 \right] \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\}, \quad (3.4)$$

where r_+ is given by (2.11). If we substitute (3.4) into (3.3), and integrate by parts, we obtain

$$\begin{aligned} u_p(x,y) = i\pi & \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\} \\ & - \frac{2\pi i \lambda_1 (\lambda_1 + \lambda_2)}{\lambda_2 - \lambda_1} e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_0^{(1)}(kR_+) d\eta \\ & - \frac{2\pi i \lambda_2 (\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2} e^{-\lambda_2 y} \int_{-\infty}^y e^{\lambda_2 \eta} H_0^{(1)}(kR_+) d\eta. \end{aligned} \quad (3.7)$$

This solution is not a regular wave function since the x -derivative is discontinuous in regions including the positive y -axis. This defect can be remedied by adding to (3.7) another solution of the wave equation as follows:

$$\begin{aligned}
 u(x,y) = i\pi \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\} \\
 - \frac{2\pi i \lambda_1 (\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2} e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_0^{(1)}(kR_+) d\eta \\
 - \frac{2\pi i \lambda_2 (\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2} e^{-\lambda_2 y} \int_{-\infty}^y e^{\lambda_2 \eta} H_0^{(1)}(kR_+) d\eta \\
 + A_1 e^{-\lambda_1 y + i\sqrt{k^2 + \lambda_1^2} |x|} + A_2 e^{-\lambda_2 y + i\sqrt{k^2 + \lambda_2^2} |x|}
 \end{aligned} \tag{3.8}$$

The introduction of the additional terms with unknown constants A_1 and A_2 now make it possible to satisfy the continuity condition (2.14). From (3.8) we obtain

$$\begin{aligned}
 \frac{\partial u}{\partial x} = i\pi \frac{\partial}{\partial x} \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\} \\
 + \frac{2\pi i k \lambda_1 (\lambda_1 + \lambda_2)}{\lambda_2 - \lambda_1} e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_1^{(1)}(kR_+) \frac{x}{R_+} d\eta \\
 + \frac{2\pi i k \lambda_2 (\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2} e^{-\lambda_2 y} \int_{-\infty}^y e^{\lambda_2 \eta} H_1^{(1)}(kR_+) \frac{x}{R_+} d\eta \\
 + iA_1 \sqrt{k^2 + \lambda_1^2} \operatorname{sgn} x e^{-\lambda_1 y + i\sqrt{k^2 + \lambda_1^2} |x|} + iA_2 \sqrt{k^2 + \lambda_2^2} \operatorname{sgn} x e^{-\lambda_2 y + i\sqrt{k^2 + \lambda_2^2} |x|}
 \end{aligned} \tag{3.9}$$

Substituting (3.9) into (2.14), and using (2.18), we obtain

$$A_1 = \frac{i\pi i \lambda_1 (\lambda_1 + \lambda_2)}{\sqrt{k^2 + \lambda_1^2} (\lambda_2 - \lambda_1)} e^{-\lambda_1 y_0} \quad (3.10)$$

and

$$A_2 = \frac{i\pi i \lambda_2 (\lambda_1 + \lambda_2)}{\sqrt{k^2 + \lambda_2^2} (\lambda_1 - \lambda_2)} e^{-\lambda_2 y_0} \quad (3.11)$$

These are the desired amplitudes of the surface waves. The exact solution of the problem is given by (3.8), where A_1 and A_2 are given by (3.10) and (3.11).

4. Three-Mode Surface Wave Structure

We next study the electromagnetic field that arises when a magnetic line dipole source is located above a plane structure that supports three surface waves. Consider the plane surface $y = 0$ and suppose a magnetic line dipole is located at the point $x = 0, y = y_0$. See Figure 1. The boundary condition on the surface is given by the impedance condition

$$\left(\frac{\partial}{\partial y} + \lambda_1\right)\left(\frac{\partial}{\partial y} + \lambda_2\right)\left(\frac{\partial}{\partial y} + \lambda_3\right)u(x,y) = 0, \quad y = 0 \quad (4.1)$$

where u is the magnetic field component H_z and λ_1, λ_2 and λ_3 are constants characteristic of the surface material. We assume $\lambda_1 \neq \lambda_2 \neq \lambda_3$. We wish to solve the time-reduced Maxwell's equations, subject to the prescribed condition, and obtain the amplitudes of the surface waves propagated along the surface. The magnetic field component u satisfies the wave equation (1.1). In addition to the prescribed boundary condition (4.1), we require the far field be outgoing at infinity and the value of the total magnetic field

(excluding the source) be finite everywhere.

Following the same procedure used in Sections 2 and 3, we let

$$v = \left(\frac{\partial}{\partial y} + \lambda_1\right)\left(\frac{\partial}{\partial y} + \lambda_2\right)\left(\frac{\partial}{\partial y} + \lambda_3\right) u. \quad (4.2)$$

Then v satisfies the equation

$$\begin{aligned} (\nabla^2 + k^2)v = -4\pi \left[-\frac{\partial^3}{\partial y_0^3} + (\lambda_1 + \lambda_2 + \lambda_3) \frac{\partial^2}{\partial y_0^2} - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) \frac{\partial}{\partial y_0} \right. \\ \left. + \lambda_1\lambda_2\lambda_3 \right] \delta(x-0)\delta(y-y_0) \end{aligned} \quad (4.3)$$

subject to the simpler conditions

$$\left. \begin{aligned} (1) \quad v &= 0, \quad y = 0 \\ (2) \quad &\text{Outgoing traveling waves at } \infty \end{aligned} \right\}. \quad (4.4)$$

A particular solution of (4.2) is

$$\begin{aligned} u_p(x, y) = \frac{e^{-\lambda_1 y}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \int_{-\infty}^y e^{\lambda_1 \eta} v(x, \eta) d\eta + \frac{e^{-\lambda_2 y}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} \int_{-\infty}^y e^{\lambda_2 \eta} v(x, \eta) d\eta \\ + \frac{e^{-\lambda_3 y}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \int_{-\infty}^y e^{\lambda_3 \eta} v(x, \eta) d\eta. \end{aligned} \quad (4.5)$$

The solution of the auxiliary problem is

$$v(x,y) = i\pi \left[-\frac{\partial^3}{\partial y_0^3} + (\lambda_1 + \lambda_2 + \lambda_3) \frac{\partial^2}{\partial y_0^2} - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) \frac{\partial}{\partial y_0} + \lambda_1\lambda_2\lambda_3 \right] \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\} \quad (4.6)$$

where r_+ is given by (2.11). If we substitute (4.6) into (4.5) and integrate by parts we obtain

$$u_p(x,y) = i\pi \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\} - \frac{2\pi i \lambda_1 (\lambda_1 + \lambda_2) (\lambda_2 + \lambda_3)}{(\lambda_2 - \lambda_1) (\lambda_3 - \lambda_1)} e^{-\lambda_1 y} \int_{-\infty}^y e^{\lambda_1 \eta} H_0^{(1)}(kR_+) d\eta - \frac{2\pi i \lambda_2 (\lambda_1 + \lambda_2) (\lambda_2 + \lambda_3)}{(\lambda_1 - \lambda_2) (\lambda_3 - \lambda_2)} e^{-\lambda_2 y} \int_{-\infty}^y e^{\lambda_2 \eta} H_0^{(1)}(kR_+) d\eta - \frac{2\pi i \lambda_3 (\lambda_1 + \lambda_3) (\lambda_2 + \lambda_3)}{(\lambda_1 - \lambda_3) (\lambda_2 - \lambda_3)} e^{-\lambda_3 y} \int_{-\infty}^y e^{\lambda_3 \eta} H_0^{(1)}(kR_+) d\eta \quad (4.7)$$

This solution is not a wave function since the x -derivative in regions including the positive y -axis is discontinuous. We can remedy this defect, however, by adding to (4.7) complementary solutions of the form

(4.8)

$$u_c(x,y) = A_1 e^{-\lambda_1 y + i \sqrt{k^2 + \lambda_1^2} |x|} + A_2 e^{-\lambda_2 y + i \sqrt{k^2 + \lambda_2^2} |x|} + A_3 e^{-\lambda_3 y + i \sqrt{k^2 + \lambda_3^2} |x|}.$$

Then

$$u(x,y) = u_p(x,y) + u_c(x,y) \quad (4.9)$$

and the combined solution (4.9) can be made to satisfy the continuity condition on $\frac{\partial u}{\partial x}$ by suitably choosing the constants A_1 , A_2 and A_3 . We now determine

$\frac{\partial u}{\partial x}$ from (4.7), (4.8) and (4.9), and substitute into (2.14). We obtain

$$A_1 = \frac{4\pi i \lambda_1 (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)}{\sqrt{k^2 + \lambda_1^2} (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} e^{-\lambda_1 y_0} \quad (4.10)$$

$$A_2 = \frac{4\pi i \lambda_2 (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)}{\sqrt{k^2 + \lambda_2^2} (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} e^{-\lambda_2 y_0} \quad (4.11)$$

$$A_3 = \frac{4\pi i \lambda_3 (\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}{\sqrt{k^2 + \lambda_3^2} (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} e^{-\lambda_3 y_0} \quad (4.12)$$

These are the desired amplitudes of the surface waves. The exact solution of the problem is now complete.

Multi-Layer Surface Wave Scattering

The solution of problems with more than three surface wave modes can be carried out using the same procedure described in Section 3 through 5. We now summarize the main steps for treating surface wave modes of any order.

The mathematical problem to be solved requires a solution of the wave equation

$$(\nabla^2 + k^2) u(x,y) = -4\pi\delta(x-0)\delta(y-y_0) \quad (2.1)$$

subject to the boundary condition

$$\prod_{j=1}^n \left(\frac{\partial}{\partial y} + \lambda_j \right) u(x,y) = 0 \quad , \quad y = 0 . \quad (2.2)$$

In addition, the far field must be outgoing and the total magnetic field (excluding the source) must be finite. We first make the substitution

$$v = \prod_{j=1}^n \left(\frac{\partial}{\partial y} + \lambda_j \right) u . \quad (2.3)$$

Then v satisfies the equation

$$(\nabla^2 + k^2) v = -4\pi \prod_{j=1}^n \left(-\frac{\partial}{\partial y_0} + \lambda_j \right) \delta(x-0)\delta(y-y_0) \quad (2.4)$$

subject to the simpler boundary conditions

$$\left. \begin{array}{l} (1) \quad v = 0 \quad , \quad y = 0 \\ (2) \quad \text{Outgoing traveling waves at } \infty \end{array} \right\} . \quad (2.5)$$

The solution for v is given by

$$v(x,y) = i\pi \prod_{j=1}^n \left(-\frac{\partial}{\partial y_0} + \lambda_j \right) \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\} . \quad (5.6)$$

A particular solution of the original field u can be found by solving (5.3).

It is

$$u_p(x,y) = \sum_{i=1}^n \prod_{\substack{j=1 \\ (i \neq j)}}^n \frac{1}{(\lambda_j - \lambda_i)} e^{-\lambda_i y} \int_{-\infty}^y e^{\lambda_i \eta} v(x,\eta) d\eta . \quad (5.7)$$

This solution is not a wave function since the x -derivative in regions including the positive y -axis is discontinuous. We can remedy this defect, however, by adding to (5.7) complementary solutions of the form

$$u_c(x,y) = \sum_{i=1}^n A_i e^{-\lambda_i y + i\sqrt{k^2 + \lambda_i^2} |x|} . \quad (5.8)$$

Then the combined solution given by

$$u(x,y) = u_p(x,y) + u_c(x,y) \quad (5.9)$$

can be made to satisfy the continuity condition on $\frac{\partial u}{\partial x}$. This is equivalent to imposing the jump condition

$$\left[\frac{\partial u}{\partial x} \right] = \lim_{x \rightarrow 0} \left\{ \frac{\partial u}{\partial x}(x > 0, y) - \frac{\partial u}{\partial x}(x < 0, y) \right\} = 0 . \quad (5.10)$$

The requirement (2.10) is sufficient to determine the constant δ_1 , which are the amplitudes of the surface waves. The exact solution for $\phi(x,y)$ will contain the sum or difference of the source and image (depending on whether or not there are an odd or even number of surface waves present, respectively), a linear combination of integrals of the type (2.1) with different λ_1 's, and of sum of surface waves as given by (2.6).

3. Determination of the Radiated Far Field

We now determine the radiated far field arising from a magnetic line dipole located above a plane surface that supports surface waves. The method is a direct and elementary procedure developed by the authors [2, 14-20] and does not require the exact results obtained in the earlier section.

Consider first the case of a structure that supports a single surface wave. The same physical configuration and notation used earlier applies.

See Figure 1. From (2.6) and (2.10) we have

$$v = \left(\frac{\partial}{\partial y} + \lambda_1 \right) u \quad (3.1)$$

and

$$v = i\pi \left(- \frac{\partial}{\partial y_0} + \lambda_1 \right) \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\} \quad (3.2)$$

Here r_- and r_+ are the distances from the source and image to the observation point, respectively, and are given by

$$r_{\pm} = \sqrt{x^2 + (y_{\pm} - y_0)^2} . \quad (6.3)$$

Let r be the distance measured from the origin. Then

$$r = \sqrt{x^2 + y^2} \quad (6.4)$$

and the far field expressions for $r_{\pm}$ become

$$r_{\pm} \approx r \pm y_0 \sin \theta , \quad (6.5)$$

provided $y_0 \ll r$. When $kr_{\pm}$ is large, (6.2) becomes

$$v(r, \theta) \approx \sqrt{\frac{8\pi}{kr}} \left[\lambda_1 \sin(ky_0 \sin \theta) - k \sin \theta \cos(ky_0 \sin \theta) \right] e^{i(kr - \frac{\pi}{4})} \quad (6.6)$$

where we have used the asymptotic expansion of the Hankel function for large argument. We expect the radiated far field of u to have a similar form, except that it will contain surface wave terms which are appreciable at large x near the surface $y = 0$. Hence, for fixed θ and large r , we have

$$u(r, \theta) \approx \frac{f(\theta)}{\sqrt{r}} e^{ikr} , \quad \theta \neq 0 \text{ or } \pi , \quad (6.7)$$

where $f(\theta)$ is unknown. Substituting (6.6) and (6.7) into (6.1), and using the result

$$\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial}{\partial x} \approx -i\theta + \frac{\partial}{\partial r} \quad (6.9)$$

we obtain $f(\theta)$ correct to terms of order $r^{-1/2}$. The asymptotic form of the far field then becomes

$$u(r, \theta) \approx \sqrt{\frac{e\pi}{kr}} \frac{[\lambda_1 \sin(ky_0 \sin \theta) - k \sin \theta \cos(ky_0 \sin \theta)]}{[ik \sin \theta + \lambda_1]} e^{-i(kr - \frac{\pi}{4})} \quad (6.10)$$

The radiation pattern of the far field is easily determined from (6.7). We

define the pattern function $F(\theta, \frac{\lambda_1}{k}, ky_0)$ to be

$$F(\theta, \frac{\lambda_1}{k}, ky_0) = \left| \sqrt{\frac{kr}{8\pi}} e^{-i(kr - \frac{\pi}{4})} u(r, \theta) \right| \quad (6.10)$$

Hence

$$F(\theta, \frac{\lambda_1}{k}, ky_0) = \frac{[\frac{\lambda_1}{k} \sin(ky_0 \sin \theta) - \sin \theta \cos(ky_0 \sin \theta)]}{[(\frac{\lambda_1}{k})^2 + \sin^2 \theta]} \quad (6.11)$$

Consider next the case of a structure that supports two surface waves. The procedure is the same as that just described. From (3.1) and (3.6) we have

$$v = (\frac{\partial}{\partial y} + \lambda_1)(\frac{\partial}{\partial y} + \lambda_2) u \quad (6.12)$$

and

$$v(x, y) = i\pi \left[\frac{\partial^2}{\partial y_0^2} - (\lambda_1 + \lambda_2) \frac{\partial}{\partial y_0} + \lambda_1 \lambda_2 \right] \left\{ H_0^{(1)}(kr_-) - H_0^{(1)}(kr_+) \right\} \quad (6.13)$$

The radiated far field for $v(r, \theta)$ is

$$v(r, \theta) \approx \sqrt{\frac{e\pi}{kr}} \left[\lambda_1 \lambda_2 \sin(ky_0 \sin \theta) - (\lambda_1 + \lambda_2) k \sin \theta \cos(ky_0 \sin \theta) - k^2 \sin^2 \theta \sin(ky_0 \sin \theta) \right] e^{i(kr - \frac{\pi}{4})} \quad (6.14)$$

We expect the radiated far field to have the form (6.7). Substituting (6.7) and (6.14) into (6.12), we obtain $f(\theta)$ correct to terms of order $r^{-1/2}$.

The expression for the radiated far field then becomes

$$u(r, \theta) \approx \sqrt{\frac{e\pi}{kr}} \frac{[(\lambda_1 \lambda_2 - k^2 \sin^2 \theta) \sin(ky_0 \sin \theta) - (\lambda_1 + \lambda_2) k \sin \theta \cos(ky_0 \sin \theta)]}{[ik \sin \theta + \lambda_1][ik \sin \theta + \lambda_2]} e^{i(kr - \frac{\pi}{4})} \quad (6.15)$$

The pattern function is given by

$$F(\theta, \frac{\lambda_1}{k}, \frac{\lambda_2}{k}, ky_0) = \frac{\left[\left(\frac{\lambda_1}{k} \frac{\lambda_2}{k} - \sin^2 \theta \right) \sin(ky_0 \sin \theta) - \left(\frac{\lambda_1}{k} + \frac{\lambda_2}{k} \right) \sin \theta \cos(ky_0 \sin \theta) \right]^2}{\left[\left(\frac{\lambda_1}{k} \right)^2 + \sin^2 \theta \right] \left[\left(\frac{\lambda_2}{k} \right)^2 + \sin^2 \theta \right]} \quad (6.16)$$

The calculation of the radiated far field for a structure that supports three surface waves can be carried out in a similar way. Using (4.1), the asymptotic form of (4.6), and (6.7) we obtain

$$u(r, \Theta) \approx \sqrt{\frac{\pi}{kr}} \left[ik \sin \Theta + \lambda_1 \right]^{-1} \left[ik \sin \Theta + \lambda_2 \right]^{-1} \left[ik \sin \Theta + \lambda_3 \right]^{-1} e^{i(kr - \frac{\pi}{2})} \\ \cdot \left[\left\{ \frac{\lambda_1 \lambda_2 \lambda_3}{k} - (\lambda_1 + \lambda_2 + \lambda_3) k \sin^2 \Theta \right\} \sin(ky_0 \sin \Theta) \right. \\ \left. - k \sin \Theta \left\{ (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) - k^2 \sin^2 \Theta \right\} \cos(ky_0 \sin \Theta) \right]. \quad (6.17)$$

The pattern function is

$$F(\Theta, \frac{\lambda_1}{k}, \frac{\lambda_2}{k}, \frac{\lambda_3}{k}, ky_0) = \left[\left(\frac{\lambda_1}{k} \right)^2 + \sin^2 \Theta \right]^{-1} \left[\left(\frac{\lambda_2}{k} \right)^2 + \sin^2 \Theta \right]^{-1} \left[\left(\frac{\lambda_3}{k} \right)^2 + \sin^2 \Theta \right]^{-1} \\ \cdot \left[\left\{ \frac{\lambda_1}{k} \frac{\lambda_2}{k} \frac{\lambda_3}{k} - \left(\frac{\lambda_1}{k} + \frac{\lambda_2}{k} + \frac{\lambda_3}{k} \right) \sin^2 \Theta \right\} \sin(ky_0 \sin \Theta) \right. \\ \left. - \left\{ \left(\frac{\lambda_1}{k} \frac{\lambda_2}{k} + \frac{\lambda_1}{k} \frac{\lambda_3}{k} + \frac{\lambda_2}{k} \frac{\lambda_3}{k} \right) - \sin^2 \Theta \right\} \sin \Theta \cos(ky_0 \sin \Theta) \right]^{-1}. \quad (6.18)$$

For structures that support more than three surface waves, the procedure for determining the radiated far field is the same. The only equations needed are (6.3), the asymptotic far field form of (6.6), and (6.7).

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